Differential transform method for solving linear and non-linear systems of partial differential equations

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1. Introduction

Systems of partial differential equations arise in many scientific fields such as solid state physics, plasmas physics, fluid dynamics, mathematical biology and chemical kinetics. A verity of methods, exact approximate and purely numerical are available for the solution of systems of partial differential equations. Most of these methods are computationally intensive because they are trial-and-error in nature, or need complicated symbolic computations. Wazwaz [1] used the Adomian decomposition method to handle some system of partial differential equations and reaction–diffusion Brusselator model. Recently, Batiha et al. [2] improved Wazwaz [3] results on the applications of variational iteration method to solve some linear and non-linear systems of partial differential equations.

In this Letter, we introduce the differential transform method as an alternative to existing methods in solving linear and non-linear systems of partial differential equations. The differential transform method is a numerical method for solving differential equations. The concept of differential transform method is first introduced by Zhou [4] in solving linear and non-linear initial value problems in electrical circuit analysis. The differential transform method obtains an analytical solution in the form of a polynomial. It is different form the traditional higher order Taylor series method, which requires symbolic computation of the necessary derivatives of the data functions. The Taylor series method is computationally taken long time for large orders. His method is well addressed in [5–11].

2. Basic idea of differential transform method

The basic definitions and fundamental operations of the two-dimensional differential transform are defined in [5–11] as follows.

Consider a function of two variable \( w(x, y) \), be analytic in the domain \( K \) and let \( (x, y) = (x_0, y_0) \) in this domain. The function \( w(x, y) \) is then represented by one series linear and non-linear systems of partial differential equations.

In this Letter, we introduce the differential transform method as an alternative to existing methods in solving linear and non-linear systems of partial differential equations. The differential transform method is first introduced by Zhou [4] in solving linear and non-linear initial value problems in electrical circuit analysis. The differential transform method obtains an analytical solution in the form of a polynomial. It is different form the traditional higher order Taylor series method, which requires symbolic computation of the necessary derivatives of the data functions. The Taylor series method is computationally taken long time for large orders. His method is well addressed in [5–11].

The differential transform method is defined as

\[
W(k, h) = \frac{1}{k!h!} \left[ \frac{\partial^{k+h} w(x, y)}{\partial x^k \partial y^h} \right] (x_0, y_0),
\]

where \( w(x, y) \) is the original function and \( W(k, h) \) is the transformed function.

The differential inverse transform of \( W(k, h) \) is defined as

\[
w(x, y) = \sum_{k=0}^{\infty} \sum_{h=0}^{\infty} W(k, h)(x-x_0)^k(y-y_0)^h.
\]
The transformed version of Eq. (6) is

\[ w(k, y) = U(k, h) \pm V(k, h) \]

The fundamental mathematical operations performed by two-dimensional differential transform method are listed in Table 1.

### 3. Applications

In this section, we shall illustrate the numerical scheme by linear and non-linear systems of partial differential equations, which have been widely discussed in the literature.

**Example 1.** We first consider the homogeneous linear system of PDEs:

\[ u_t - v_x + (u + v) = 0, \]
\[ v_t - u_x + (u + v) = 0, \]

with the initial conditions

\[ u(x, 0) = \sinh x, \quad \text{and} \quad v(x, 0) = \cosh x. \]

The transformed version of (4) and (5) are

\[ (h + 1)U(k, h + 1) - (k + 1)V(k + 1, h) + U(k, h) + V(k, h) = 0, \quad \text{with} \quad k = 0, 1, 2, 3, \ldots. \]

Substituting (9) and (10) in (7) and (8), we obtained the following closed form solutions

\[ u(x, t) = \sum_{k=0}^{\infty} \sum_{h=0}^{\infty} U(k, h)x^k t^h = \left( \sum_{k=0}^{\infty} \frac{1}{k!}x^k \right) \left( \sum_{h=0}^{\infty} \frac{1}{h!}t^h \right) = \cosh(x - t), \]

\[ v(x, t) = \sum_{k=0}^{\infty} \sum_{h=0}^{\infty} V(k, h)x^k t^h = 1 + \left( \sum_{h=0}^{\infty} \frac{1}{h!}t^h \right) = 1 + e^{x-t}, \]

which are the exact solutions of (4)–(6).

**Example 2.** Now we consider the following non-homogeneous linear system:

\[ u_t - v_x - (u - v) = -2, \]
\[ v_t + u_x - (u - v) = -2, \]

with the initial conditions

\[ u(x, 0) = 1 + e^x, \quad \text{and} \quad v(x, 0) = -1 + e^x. \]

The transformed version of (13) and (14) are

\[ (h + 1)U(k, h + 1) - (k + 1)V(k + 1, h) - U(k, h) + V(k, h) = -2, \quad \text{and} \quad \text{with} \quad k = 0, 1, 2, 3, \ldots. \]

Substituting (19) and (18) in (16) and (17), we obtained the following closed form solutions:

\[ u(x, t) = \sum_{k=0}^{\infty} \sum_{h=0}^{\infty} U(k, h)x^k t^h = 1 + \left( \sum_{k=0}^{\infty} \frac{1}{k!}x^k \right) \left( \sum_{h=0}^{\infty} \frac{1}{h!}t^h \right) = 1 + e^{x-t}, \]
\[ v(x, t) = \sum_{k=0}^{\infty} \sum_{h=0}^{\infty} V(k, h)x^k t^h = -1 + \left( \sum_{k=0}^{\infty} \frac{1}{k!}x^k \right) \left( \sum_{h=0}^{\infty} \frac{(-1)^h}{h!}t^h \right) = -1 + e^{x-t}, \]

which are the exact solutions of (13)–(15).
Example 3. We consider the coupled Burgers’ equations [12]:

\[ u_t - u_{xx} - 2uu_x + (uv)_x = 0, \quad (22) \]
\[ v_t - v_{xx} - 2vv_x + (uv)_x = 0, \quad (23) \]

with the initial conditions

\[ u(x, 0) = \sin x, \quad \text{and} \quad v(x, 0) = \sin x. \]

The transformed version of (22) and (23) are

\[
(h + 1)U(k, h + 1) - (k + 1)(k + 2)U(k + 2, h) - 2 \sum_{r=0}^{k} \sum_{s=0}^{h} U(r, h - s)(k - r + 1)U(k - r + 1, s)
\]
\[
+ \sum_{r=0}^{k} h (r + 1)U(r + 1, h - s)V(k - r, s)
\]
\[
+ \sum_{r=0}^{k} \sum_{s=0}^{h} (r + 1)V(r + 1, h - s)U(k - r, s) = 0,
\]

\[
(h + 1)V(k, h + 1) - (k + 1)(k + 2)V(k + 2, h) - 2 \sum_{r=0}^{k} \sum_{s=0}^{h} V(r, h - s)(k - r + 1)V(k - r + 1, s)
\]
\[
+ \sum_{r=0}^{k} h (r + 1)V(r + 1, h - s)V(k - r, s)
\]
\[
+ \sum_{r=0}^{k} \sum_{s=0}^{h} (r + 1)V(r + 1, h - s)U(k - r, s) = 0. \quad (24) \]

The transformed version of Eq. (24) is

\[ U(k, 0) = \begin{cases} 
0, & k = 0, 2, 4, \ldots, \\
\frac{1}{k!}, & k = 1, 5, \ldots, \\
-\frac{1}{k!}, & k = 3, 7, \ldots, 
\end{cases} \quad (27) \]

\[ V(k, 0) = \begin{cases} 
0, & k = 0, 2, 4, \ldots, \\
\frac{1}{k!}, & k = 1, 5, \ldots, \\
-\frac{1}{k!}, & k = 3, 7, \ldots, 
\end{cases} \quad (28) \]

substituting (27) and (28) in (25) and (26), we obtained the following closed form solutions:

\[ u(x, t) = \sum_{k=0}^{\infty} \sum_{h=0}^{\infty} U(k, h)x^k t^h = \left( \frac{x}{1!} - \frac{x^3}{3!} + \cdots \right) \left( 1 - \frac{t^2}{2!} - \cdots \right), \quad (29) \]
\[ v(x, t) = \sum_{k=0}^{\infty} \sum_{h=0}^{\infty} V(k, h)x^k t^h = \left( \frac{x}{1!} - \frac{x^3}{3!} + \cdots \right) \left( 1 - \frac{t^2}{2!} - \cdots \right), \quad (30) \]

which are the exact solutions of (22)–(24).

4. Conclusion

In this work, differential transform method is extended to solve the linear and non-linear systems of partial differential equations. The present study has confirmed that the differential transform method offers significant advantages in terms of its straightforward applicability, its computational effectiveness and its accuracy.

References